Int. J. Solids Structures Vol. 16, pp. 445–450
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0020-7683/80/0501-0445/$02.00/0

BUCKLING OF MASONRY PIER UNDER ITS OWN WEIGHT

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(*Received* 16 *January* 1979; *in revised form* 11 *June* 1979)

Abstract—The stability of a column made of no-tension (or low tension) material and buckling under its own weight is analysed. Piers of rectangular cross section are considered and load-deflection curves are derived in closed form.

NOTATION

b	width of pier
d	depth of pier
d_c	$0.72-0.29\alpha d/\delta$
d_δ	$1.14d/\delta-0.94$
d_t	$\tan^{-1}(1.14\sqrt{d_c}/\sqrt{d_\delta})+\tan^{-1}(0.16/\sqrt{d_\delta})$
m	$\sqrt{(q/EI)}$
q	weight per unit height of column
r_x	distance of line of action of W_x from x axis
$t(y)$	$\int_0^y J_{-1/3}(\varphi)\,d\varphi$, approximately
v	distance of line of action of W_x from centreline of column
x, w	coordinates
x_c	length of cracked zone
y	$2m(L-x)^{3/2}/3$
A	$2q/9Eb$
EI	bending rigidity of uncracked section
F	function of d/δ and $2mL^{3/2}/3$ defined in eqn (13)
H	function of d/δ defined in eqn (11)
L	height of column
$J_{-1/3}(\varphi)$	Bessel function of order $-1/3$ of the first kind
W_x	weight of column above x
α	parameter, measure of tensile strength defined in eqns (9a) and (9b)
δ	deflection of top of column
σ_t, $\sigma_{0,av}$	tensile strength, qL/bd
σ_{conv}	tensile stress on convex face

1. INTRODUCTION

Analytical investigations of masonry columns usually assume that absence of a substantial tensile strength of the material permits the cracking of the cross section of the column thus decreasing the effective depth of that section. This leads to the concept of a column having a varying moment of inertia, but the variation is not known, is not symmetric about the centreline, and varies with the deflection.

If the column is made of low strength mortar its tensile strength will be nearly zero ($\alpha = 1$); cracking will occur at the onset of tension, leading to a triangular stress distribution in the cracked zone (Fig. 2). If the column possesses a small amount of tensile strength ($\alpha = 1$), such tension will build up at the convex face resulting in a trapezoidal stress distribution (Figs. 1 and 2). The same considerations apply at the base where the column is assumed to be fixed and cracking may, or may not, occur.

Columns of this kind have attracted some interest in the past. Angervo[1], Chapman and Slatford[2], Yokel[3], Risager[4], and Frisch-Fay[5] have considered columns under one concentrated axial force. Bo-Goeran Hellers[6], Sahlin[7], Yokel, Mathey and Dikkers[8], and Frisch-Fay[9, 10] included transverse loading in addition to the axial load. It appears that the only investigators to consider the effect of the self weight were Tesfaye and Broome[11]. Their investigation assumed that all column sections are cracked and that the centroid of the buckled column above x will lie on the centreline of the deflected shape; in reality, it will lie off the centreline and towards the concave direction (Fig. 1).

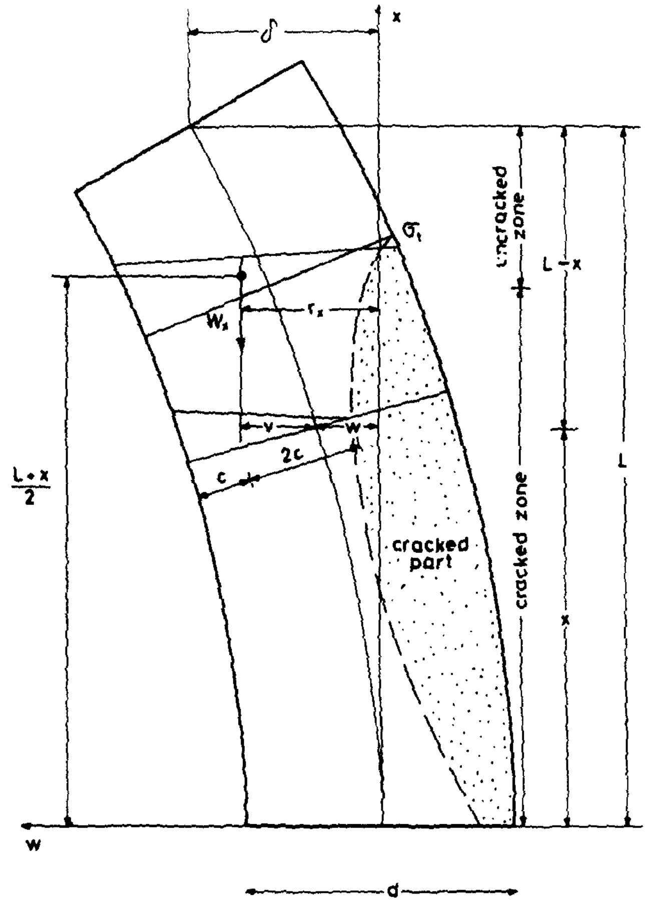

Fig. 1. Deflected shape of pier.

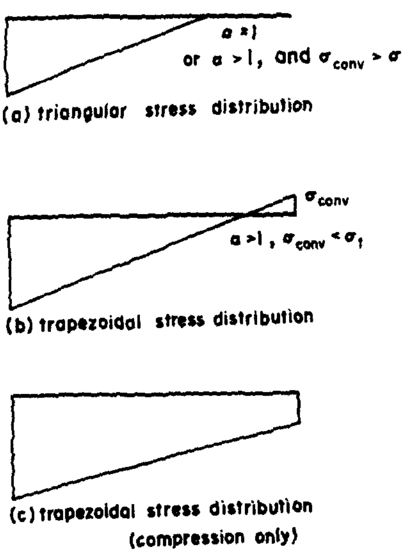

Fig. 2. Possible stress distributions across the depth of the pier.

The objective of the present discussion is to consider the effect of the self weight on the stability of a masonry column. Both cracked and uncracked zones will be considered and the centroid of the deflected column will be placed off centreline.

2. FORMULATION OF THE PROBLEM

The weight of a rectangular column of dimensions $d \times b$ above x is $W_x = q(L-x)$ and acts at a distance r_x from the x axis (Fig. 1). As seen,

$$r_x = \int_x^L w\,dx/(L-x), \text{ and } v = \int_x^L w\,dx/(L-x) - w,$$

where v is the distance of the line of action of W_x from the deflected centreline of the column.

According to [5] the curvature of the cracked part of a rectangular column is governed by the differential equation

$$d^2w/dx^2 - 2q(L-x)/9Eb(1/2d - v)^2 = 0. \quad (1)$$

For the purpose of establishing $v = \theta(x)$ the buckled shape of an uncracked column will be assumed. The slope of such a column buckling under its own weight is

$$dw/dx = C\sqrt{(L-x)}J_{-1/3}[2m(L-x)^{3/2}/3] \quad [12]$$

where $m = \sqrt{(q/EI)}$.
Then,

$$w = C\int \sqrt{(L-x)}J_{-1/3}[2m(L-x)^{3/2}/3]\,dx + D.$$

With the substitution $y = 2m(L-x)^{3/2}$ and satisfying the boundary conditions $w|_{y=0} = \delta$, and $w|_{2mL^{3/2}/3} = 0$,
we evaluate C and D to get

$$w = \delta\left[1 - \int_0^y J_{-1/3}(\varphi)\,d\varphi \Big/ \int_0^{1.87} J_{-1/3}(\varphi)\,d\varphi\right] \quad (2)$$

because $2mL^{3/2}/3 = 1.87$ when buckling occurs.

The integral $\int_0^y J_{-1/3}(\varphi)\,d\varphi$ has been tabulated in [13] whence the denominator in eqn (2) is 1.52. Approximation of $\int_0^y J_{-1/3}(\varphi)\,d\varphi$ between $0 \le y \le 2.1$ by a seventh order parabola changes eqn (2) into

$$w = \delta[1 - 0.658t(y)] \quad (3)$$

where

$$t(y) = 0.1263y^7 - 1.075y^6 + 3.7848y^5 - 7.1212y^4 + 7.7595y^3 - 5.3379y^2 + 3.1344y. \quad (4)$$

v in eqn (1) can now be evaluated from

$$v = r_x - w = \delta\left[\frac{x|_x^L}{L-x} - \frac{0.658\int_x^L t(y)\,dx}{L(1-x/L)} - 1 + 0.658t(y)\right]. \quad (5)$$

Remembering that $y = 2mL^{3/2}(1-x/L)^{3/2}/3$, v can now be reduced to the form

$$v/\delta = f(1-x/L). \quad (6)$$

The following numerical values are obtained from eqn (6) in the interval $0 \le x/L \le 1$,

x/L	0	0.2	0.4	0.6	0.8	1.0
v/δ	0.3894	0.4255	0.3802	0.2823	0.1500	0

For ease of solving eqn (1) these values of v/δ will be approximated by a best fitting second order parabola based on least squares,

$$v = \delta(-0.571x^2/L^2 + 0.16x/L + 0.4) \tag{7}$$

3. DERIVATION OF THE LOAD-DEFLECTION RELATIONSHIP

Defining $A = 2q/9Eb$, substitution of eqn (7) into eqn (1), integration of eqn (1), and satisfying of the boundary condition $\mathrm{d}w/\mathrm{d}x|_{x=0} = 0$ leads to

$$\frac{1}{A}\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{L^2}{\delta^2}\left[\frac{0.98x/L + d/\delta - 0.96}{0.571x^2/L^2 - 0.16x/L + 1/2d/\delta - 0.4} - \frac{d/\delta - 0.96}{1/2d/\delta - 0.4} + \frac{1.96}{\sqrt{d_\delta}}\left[\tan^{-1}\left(\frac{1.14x/L - 0.16}{\sqrt{d_\delta}}\right) + \tan^{-1}\left(\frac{0.16}{\sqrt{d_\delta}}\right)\right]\right] \Big/ d_\delta \tag{8}$$

where $d_\delta = 1.14d/\delta - 0.94$ and $\delta/d \le 1.21$. Equation (8) is the slope of the cracked part of the column.

The length of the cracked zone is x_c. This zone extends from $v = \alpha d/6$, α being a measure of the tensile strength σ_t of the material. Equating this value of v with the r.h.s. of eqn (7) yields

$$x_c/L = 0.14 + \sqrt{(0.72 - 0.29\alpha d/\delta)} \tag{9}$$

for $\delta/d \ge 0.405\alpha$. The parameter α is defined as

$$\alpha = 1 + \frac{|\sigma_t|}{|W_{xc}/bd|}$$

in particular $\alpha = 1$, when $\sigma_t = 0$, (9a)

$$\alpha > 1, \text{ when } |\sigma_t| > 0.$$

Since $W_{xc} = q(L - x_c)$, α must satisfy

$$\alpha = 1 + \frac{|\sigma_t/\sigma_{0,av}|}{0.86 - \sqrt{(0.72 - 0.29\alpha d/\delta)}}$$

$$\sigma_{0,av} = qL/bd. \tag{9b}$$

We note here that x_c/L has two solutions. The solution of x_c/L with the negative square root is valid only for $\delta/d \le 0.414\alpha$ and indicates that in the restricted interval $0.414\alpha \ge \delta/d \ge 0.405\alpha$ the column is uncracked for a certain length above the base; above this zone is a cracked part of non-dimensional length $2\sqrt{(0.72 - 0.29\alpha d/\delta)}$, with an uncracked zone at the top of the pier. Because of the very small range of values of δ/d for which this can happen the first solution of x_c/L as shown in eqn (9) will only be considered here.

The slope of the uncracked zone is

$$\mathrm{d}w/\mathrm{d}x = \sqrt{L}\sqrt{(1 - x/L)}C_2J_{-1/3}[2m(L - x)^{3/2}/3] \tag{10}$$ [12]

C_2 is found by realizing that at $v = \alpha d/6$ the slopes in the cracked and uncracked parts are identical. Thus, equating eqns (8) and (9) when $v = \alpha d/6$ we find the slope within the uncracked zone as

$$\mathrm{d}w/\mathrm{d}x = A\frac{L^2}{\delta^2 d_\delta}H(d/\delta)\sqrt{(1 - x/L)} \times \frac{J_{-1/3}[2mL^{3/2}(1 - x/L)^{3/2}/3]}{J_{-1/3}[2mL^{3/2}(0.86 - \sqrt{d_c})^{3/2}/3](0.86 - \sqrt{d_c})^{1/2}} \tag{11}$$

where $d_c = 0.72 - 0.29\alpha d/\delta$

$$H(d/\delta) = \frac{1 - 0.82\frac{\delta}{d} + 0.98\frac{\delta}{d}\sqrt{d_c}}{1/2 - \alpha/6} - \frac{d/\delta - 0.96}{1/2d/\delta - 0.4} + \frac{1.96}{\sqrt{d_\delta}}d_t$$

and

$$d_t = \tan^{-1}\left(\frac{1.14\sqrt{d_c}}{\sqrt{d_\delta}}\right) + \tan^{-1}\left(\frac{0.16}{\sqrt{d_\delta}}\right).$$

The deflection of the cracked part can be obtained by integrating eqn (8). This leads to

$$w = A\frac{L^3}{\delta^2 d_\delta}\left[\frac{2d/\delta - 1.92 + 1.96x/L}{\sqrt{d_\delta}}\tan^{-1}\left(\frac{1.14x/L - 0.16}{\sqrt{d_\delta}}\right)\right.$$

$$\left. - \frac{d/\delta - 0.96}{1/2d/\delta - 0.4}\frac{x}{L} + \frac{1.96}{\sqrt{d_\delta}}\frac{x}{L}\quad\tan^{-1}\left(\frac{0.16}{\sqrt{d_\delta}}\right) + 0.86\ln\frac{d_\delta}{2.28}\right] + C_3. \tag{12}$$

The deflection of the uncracked zone can be found by integrating eqn (11) and from this,

$$w = F(d/\delta)\int \sqrt{(1 - x/L)}J_{-1/3}[2mL^{3/2}(1 - x/L)^{3/2}/3]\,dx + C_4 \tag{13}$$

where

$$F = A\frac{L^2}{\delta^2 d_\delta}\frac{H}{(0.86 - \sqrt{d_c})^{1/2}J_{-1/3}[2mL^{3/2}(0.86 - \sqrt{d_c})^{3/2}/3]}.$$

Recalling that $y = 2mL^{3/2}(1 - x/L)^{3/2}/3$ eqn (13) reduces to

$$w = \delta - \frac{F}{m\sqrt{L}}t(y) \tag{14}$$

since $w|_{y=0}^{x=L} = \delta$, and $t = \int_0^y J_{-1/3}(\varphi)$ vanishes at $y = 0$. The function $t(y)$ has been defined in eqn (4).

The undetermined constant C_3 in eqn (12) can be evaluated from the requirement that the deflection at $v = \alpha d/6$ is the same in the cracked and uncracked zones. This changes eqn (12) into

$$\begin{aligned} w = \delta &- \frac{F}{m\sqrt{L}}t[2mL^{3/2}(0.86 - \sqrt{d_c})^{3/2}/3] \\ &+ A\frac{L^3}{\delta^2 d_\delta}\left[\frac{2d/\delta + 1.96x/L - 1.92}{\sqrt{d_\delta}}\tan^{-1}\left(\frac{1.14x/L - 0.16}{\sqrt{d_\delta}}\right)\right. \\ &- \frac{d/\delta - 0.96}{1/2d/\delta - 0.4}(x/L - 0.14 - \sqrt{d_c}) + \frac{1.96x/L - 0.274 - 1.96\sqrt{d_c}}{\sqrt{d_\delta}}\tan^{-1}\left(\frac{0.16}{\sqrt{d_\delta}}\right) \\ &\left.+ \frac{1.645 - 2d/\delta - 1.96\sqrt{d_c}}{\sqrt{d_\delta}}\tan^{-1}\left(\frac{1.14\sqrt{d_c}}{\sqrt{d_\delta}}\right)\right]. \end{aligned} \tag{15}$$

Equation (15) is the deflection of the cracked part of the column. The load-deflection relationship, $mL^{3/2}$ v. d/δ is found from eqn (15) by setting $w|_{x=0} = 0$. We note here that

$$\frac{F}{m\sqrt{L}} = \frac{2q}{9Eb}\frac{L^2}{\delta^2}\frac{1}{m\sqrt{L}}\times(\text{other terms}) = \frac{1}{54}\frac{d^2}{\delta^2}\sqrt{(q/EI)}L^{3/2}\times(\text{other terms})$$

and that $AL^3/\delta^2 = d^2qL^3/54\delta^2EI$. Thus, from eqn (15),

$$\frac{\delta}{d} - \frac{1}{54}mL^{3/2}\frac{d^2}{\delta^2 d_\delta}t[2mL^{3/2}(0.86 - \sqrt{d_c})^{3/2}/3]\frac{H}{(0.86 - \sqrt{d_c})^{1/2}J_{-1/3}[2mL^{3/2}(0.86 - \sqrt{d_c})^{3/2}/3]}$$

$$+ \frac{1}{54}(mL^{3/2})^2\frac{d^2}{\delta^2 d_\delta}\frac{1.645 - 2d/\delta - 1.96\sqrt{d_c}}{\sqrt{d_\delta}}d_t + \frac{d/\delta - 0.96}{1/2d/\delta - 0.4}(0.14 + \sqrt{d_c}) = 0. \tag{16}$$

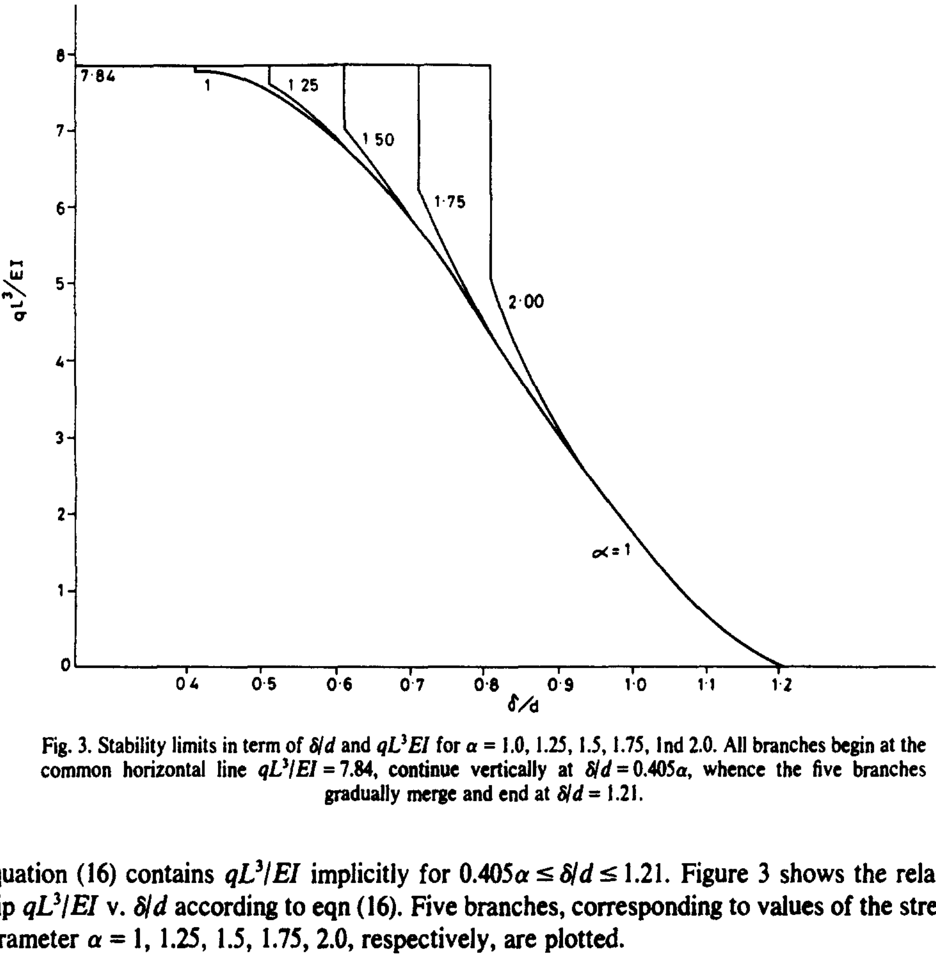

Fig. 3. Stability limits in term of δ/d and qL^3EI for α = 1.0, 1.25, 1.5, 1.75, Ind 2.0. All branches begin at the common horizontal line $qL^3/EI = 7.84$, continue vertically at $\delta/d = 0.405\alpha$, whence the five branches gradually merge and end at $\delta/d = 1.21$.

Equation (16) contains qL^3/EI implicitly for $0.405\alpha \leq \delta/d \leq 1.21$. Figure 3 shows the relationship qL^3/EI v. δ/d according to eqn (16). Five branches, corresponding to values of the strength parameter α = 1, 1.25, 1.5, 1.75, 2.0, respectively, are plotted.

4. CONCLUSIONS

The graph in Fig. 3 and the underlying analysis show that all columns, including those with a reasonable tensile strength, lose their stability if the ratio $\delta/d \geq 1.21$. The graph also shows that below $(\delta/d)_{min} = 0.405\alpha$ there is no difference between a conventional column and a pier of a material with zero, or near zero, tensile strength. At the critical deflection $\delta/d = 0.405\alpha$ the stability limit of a no-tension material column ($qL^3/EI = 7.79$) is only about 1% below that of a conventional column ($qL^3/EI = 7.84$).

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